

AROUND QUILLEN'S THEOREM A

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ABSTRACT. We reformulate some exact sequences of Quillen into a spectral sequence converging to the homology of certain K -theory spaces.

In the memory of Daniel Quillen

INTRODUCTION

Let A be a Dedekind domain. Inspection shows immediately that the exact sequences of [10, Th. 3], used by Quillen to prove that the K -groups of A are finitely generated when A is a ring of S -integers in a global field, assemble to define an exact couple, hence a spectral sequence converging to the homology of KA . This gives potentially more power to Quillen's method, which as such yields no information on the ranks of these K -groups.

A natural way to imagine such a spectral sequence is to consider the maps

$$BQ_{n-1}\mathcal{P}(A) \rightarrow BQ_n\mathcal{P}(A)$$

used by Quillen as *homotopy cofibrations* rather than homotopy fibrations. That the resulting rank spectral sequence coincides with the above-mentioned spectral sequence follows from a simple argument of Vogel, see Remark 3.3.4.

The aim of this note is to construct the rank spectral sequence in a way as functorial as possible. The two operative ingredients are Thomason's theorem on the nerve of a Grothendieck construction (Theorem 1.4.3) and the notion of *cellular functor* (Definition 2.3.2), which is well-adapted to the present context thanks to Proposition 2.3.4. The main theorem is Theorem 3.3.3: we apply the theory in the slightly more general case of torsion-free sheaves over an integral scheme, which might (or might not) be useful elsewhere.

The next step would be to describe, or compute, the d^1 differentials of this rank spectral sequence. This appears quite tricky and I don't tackle this issue here.

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I don’t think Theorem A is used explicitly anywhere. Yet I feel its spirit is prevalent in this text, hence the title.

Notation. We denote by **Set**, **Ab**, **sSet**, **Cat** the category of (small) sets, abelian groups, simplicial sets, categories. For $n \geq 0$, $[n]$ denotes the totally ordered set $\{0, \dots, n\}$, considered as a small category. We write $*$ for the category with one object and one morphism (sometimes for the set with 1 element). Finally, Δ denotes the category of simplices (objects: finite nonempty ordinals, morphisms: non-decreasing maps).

We shall use Mac Lane’s comma notation [8, p. 47]: if we have a diagram of functors

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xleftarrow{F'} \mathcal{C}'$$

the comma category $F \downarrow F'$ has for objects the diagram $F(c) \rightarrow F'(c')$, and for morphisms the obvious commutative diagrams. We use the following abbreviations: if $c \in \mathcal{C}$, yielding the functor $F_c : * \rightarrow \mathcal{C}$, we write $F_c \downarrow F' = c \downarrow F'$; similarly on the right.

1. NERVES WITH COEFFICIENTS

Subsections 1.1–1.3 can essentially be found in Goerss-Jardine [5, Ch. IV].

1.1. Set-valued coefficients.

1.1.1. Let $\mathcal{D} \in \mathbf{Cat}$ be a small category. The *nerve* of \mathcal{D} is the simplicial set $N(\mathcal{D})$ with

$$N_n(\mathcal{D}) = Ob \mathbf{Cat}([n], \mathcal{D}) = \coprod_{d_0 \rightarrow \dots \rightarrow d_n} *, \quad d_i \in \mathcal{D}$$

cf. [4, II, 4.1].

1.1.2. Let $\mathcal{D} \in \mathbf{Cat}$ and let $F : \mathcal{D} \rightarrow \mathbf{Set}$ be a covariant functor. The *nerve of \mathcal{D} with coefficients in F* is the simplicial set $N(\mathcal{D}, F)$ with

$$N_n(\mathcal{D}, F) = \coprod_{d_0 \rightarrow \dots \rightarrow d_n} F(d_0), \quad d_i \in \mathcal{D}$$

cf. [4, App. II, 3.2]. For F the constant functor with value $*$, we recover the nerve of \mathcal{D} .

1.1.3. Let (\mathcal{D}, F) be as in 1.1.2. We have the associated category

$$[\mathcal{D}, F] = \{(d, x) \mid d \in \mathcal{D}, x \in F(d)\}$$

where a morphism $(d, x) \rightarrow (d', x')$ is a morphism $f \in \mathcal{D}(d, d')$ such that $F(f)(x) = x'$. This category has two other equivalent descriptions:

- (1) [4, II, 1.1] Let $y = \mathcal{D}^{op} \rightarrow \mathbf{Cat}(\mathcal{D}, \mathbf{Set})$ be the “coYoneda” embedding: then $[\mathcal{D}, F] \simeq y \downarrow F$, where F is considered as an object of $\mathbf{Cat}(\mathcal{D}, \mathbf{Set})$.
- (2) $[\mathcal{D}, F] \simeq * \downarrow F$, where $*$ $\in \mathbf{Cat}(\mathcal{D}, \mathbf{Set})$ is the constant functor with value $*$ and F is considered as a functor.

The following lemma is obvious:

1.1.4. **Lemma.** *There is a canonical isomorphism: $N(\mathcal{D}, F) \simeq N([\mathcal{D}, F])$.*

1.2. Abelian group-valued coefficients.

1.2.1. Suppose that F takes its values in the category \mathbf{Ab} of abelian groups. Then $N(\mathcal{D}, F)$ is a simplicial abelian group; it has *homology groups* [4, App. II, 3.2, p. 153]

$$H_i(\mathcal{D}, F) = H_i([N(\mathcal{D}, F)]) = \pi_i(N(\mathcal{D}, F); 0)$$

where $[X]$ is the chain complex associated to a simplicial abelian group X by taking for differentials the alternating sums of the faces.

1.2.2. Let us recall the Eilenberg-Zilber–Cartier theorem as expounded in [6, p. 7] (see also [3, 2.9 and 2.16]). To a bisimplicial abelian group X , one may associate a double complex $[X]$ as in the simplicial case. To a bisimplicial object X one may associate the diagonal simplicial object δX :

$$(\delta X)_n = X_{n,n}$$

and to a double complex C one may associate the total complex $\text{Tot } C$:

$$(\text{Tot } C)_n = \bigoplus_{p+q=n} C_{p,q}.$$

Then, given the diagram of functors

$$\begin{array}{ccc} (\Delta \times \Delta)^{op} \mathbf{Ab} & \xrightarrow{[\]} & C_{++}(\mathbf{Ab}) \\ \delta \downarrow & & \text{Tot} \downarrow \\ \Delta^{op} \mathbf{Ab} & \xrightarrow{[\]} & C_+(\mathbf{Ab}) \end{array}$$

there exist two natural transformations

$$\begin{aligned} \text{shuffle}_X : \text{Tot}[X] &\rightarrow [\delta X] \\ \text{Alexander-Whitney}_X : [\delta X] &\rightarrow \text{Tot}[X] \end{aligned}$$

which are quasi-inverse homotopy equivalences.

1.3. Simplicial set-valued coefficients.

1.3.1. If X is a simplicial set, we may associate to it the free simplicial abelian group $\mathbf{Z}X$ generated by X , with $(\mathbf{Z}X)_n = \mathbf{Z}X_n$. Similarly with a bisimplicial set. The *homology* of X is the homotopy of $\mathbf{Z}X$, or equivalently the homology of $[\mathbf{Z}X]$. Similarly with coefficients in an abelian group A , using $\mathbf{Z}X \otimes A$.

We shall usually write $[\mathbf{Z}X] =: C_*(X)$; if $X = N(\mathcal{D})$ for a category \mathcal{D} , we abbreviate $C_*(X)$ into $C_*(\mathcal{D})$.

1.3.2. Let $\mathcal{D} \in \mathbf{Cat}$ and let $\mathbf{F} : \mathcal{D} \rightarrow \mathbf{sSet}$ be a functor. We may generalise the definition of 1.1.2 to get a *bisimplicial set* $N(\mathcal{D}, \mathbf{F})$:

$$N_{p,q}(\mathcal{D}, \mathbf{F}) = \coprod_{d_0 \rightarrow \dots \rightarrow d_p} \mathbf{F}_q(d_0), \quad d_i \in \mathcal{D}.$$

We may then take the diagonal $\delta N(\mathcal{D}, \mathbf{F})$, which is a simplicial set. We define

$$\begin{aligned} \pi_i(\mathcal{D}, \mathbf{F}; (d_0, x_0)) &= \pi_i(\delta N(\mathcal{D}, \mathbf{F}); (d_0, x_0)) \\ C_*(\mathcal{D}, \mathbf{F}; A) &= C_*(\delta N(\mathcal{D}, \mathbf{F})) \otimes A \\ H_i(\mathcal{D}, \mathbf{F}; A) &= H_i(C_*(\mathcal{D}, \mathbf{F}; A)) \end{aligned}$$

for $d_0 \in \mathcal{D}_0$ and $x_0 \in F_0(d_0)$ a chosen base point, and for A an abelian group of coefficients.

1.3.3. **Lemma.** *a) Let $X = (X_{p,q}), Y = (Y_{p,q})$ be two bisimplicial sets and $\varphi : X \rightarrow Y$ a bisimplicial map. Suppose that, for each $p \geq 0$, $\varphi_{p,*} : X_{p,*} \rightarrow Y_{p,*}$ is a weak equivalence. Then $\delta\varphi : \delta X \rightarrow \delta Y$ is a weak equivalence.*

b) Let $\mathbf{F}, \mathbf{G} : \mathcal{D} \rightarrow \mathbf{sSet}$ be two functors, and let $\varphi : \mathbf{F} \rightarrow \mathbf{G}$ be a morphism of functors. Suppose that, for each $d \in \mathcal{D}$, $\varphi(d) : \mathbf{F}(d) \rightarrow \mathbf{G}(d)$ is a weak equivalence. Then $\delta N(\mathcal{D}, \varphi) : \delta N(\mathcal{D}, \mathbf{F}) \rightarrow \delta N(\mathcal{D}, \mathbf{G})$ is a weak equivalence.

Proof. a) is well-known (for example, it is the special case of the Bousfield-Friedlander theorem of [5, Th. 4.9 p. 229] where $Y = W = *$), and b) follows from a). \square

1.3.4. For an abelian group A and for $q \geq 0$, we may consider the additive functor

$$\begin{aligned} H_q(\mathbf{F}, A) : \mathcal{D} &\rightarrow \mathbf{Ab} \\ d &\mapsto H_q(\mathbf{F}(d), A). \end{aligned}$$

This gives a meaning to:

1.3.5. **Lemma.** *There is a spectral sequence*

$$E_{p,q}^2 = H_p(\mathcal{D}, H_q(\mathbf{F}, A)) \Rightarrow H_{p+q}(\mathcal{D}, \mathbf{F}; A)$$

Proof. We shall use 1.2.2: it implies that $[\delta \mathbf{Z}N(\mathcal{D}, \mathbf{F})]$ is homotopy equivalent to $\text{Tot}[\mathbf{Z}N(\mathcal{D}, \mathbf{F})]$. Therefore

$$\begin{aligned} H_*(\mathcal{D}, \mathbf{F}; A) &:= H_*([\delta \mathbf{Z}N(\mathcal{D}, \mathbf{F})] \otimes A) \\ &\simeq H_*(\text{Tot}[\mathbf{Z}N(\mathcal{D}, \mathbf{F})] \otimes A) = H_*(\text{Tot} \bigoplus_{p,q} \bigoplus_{c_0 \rightarrow \dots \rightarrow c_p} [\mathbf{Z}F_q(d_0)] \otimes A). \end{aligned}$$

Consider the first spectral sequence associated to this double complex in Cartan-Eilenberg [2, Ch. XV, §6]: the formula (1) of loc. cit, p. 331 shows that it is the desired spectral sequence. \square

1.4. Category-valued coefficients.

1.4.1. Let $\mathcal{D} \in \mathbf{Cat}$ and let $\mathbf{F} : \mathcal{D} \rightarrow \mathbf{Cat}$ be a functor. Composing \mathbf{F} with the nerve functor, we get a functor $N \circ \mathbf{F} : \mathcal{D} \rightarrow \mathbf{sSet}$, hence a bisimplicial set as in 1.3.2

$$N(\mathcal{D}, \mathbf{F}) = N(\mathcal{D}, N(\mathbf{F})).$$

1.4.2. We now extend the construction in 1.1.3. This yields the category $\mathcal{D} \int \mathbf{F}$ (Grothendieck construction, [SGA1, Exp. VI, §§8,9]):

- Objects are pairs (d, x) , $d \in \text{Ob}(\mathcal{D})$, $x \in \text{Ob}(\mathbf{F}(d))$.
- For two objects $(d, x), (d', x')$, a morphism $(d, x) \rightarrow (d', x')$ is a morphism $f : d \rightarrow d'$ and a morphism $g : \mathbf{F}(f)(x) \rightarrow x'$.
- For three objects $(d, x), (d', x'), (d'', x'')$ and two morphisms $(f, g) : (d, x) \rightarrow (d', x')$, $(f', g') : (d', x') \rightarrow (d'', x'')$, $(f', g') \circ (f, g) := (f' \circ f, g' \circ \mathbf{F}(f')(g))$.

The Grothendieck construction is covariant in \mathbf{F} . In particular, there is a canonical functor $\mathcal{D} \int \mathbf{F} \rightarrow \mathcal{D}$ induced by the morphism $\mathbf{F} \rightarrow *$, where $*$ is the constant functor with value the point category. We call this functor the *augmentation*.

Lemma 1.1.4 then generalises as

1.4.3. **Theorem** (Thomason [11, Th. 1-2]). *There is a canonical weak equivalence*

$$\delta N(\mathcal{D}, \mathbf{F}) \rightarrow N(\mathcal{D} \int \mathbf{F})$$

sending a cell $(d_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} d_n, x_0 \xrightarrow{g_1} \dots \xrightarrow{g_n} x_n) \ (x_i \in F(d_i))$ to the cell

$$(d_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} d_n, F(f_n \dots f_1)y_0 \xrightarrow{h_1} \dots \xrightarrow{h_n} y_n) \ (y_i \in F(d_i))$$

with $y_i := F(f_i \dots f_1)x_i$ and $h_i = F(f_i \dots f_1)g_i$. \square

1.4.4. Let $T : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between two small categories. To T we associate the functor $\mathbf{F}_T : \mathcal{D} \rightarrow \mathbf{Cat}$ sending d to $T \downarrow d$. The category $\mathcal{D} \int \mathbf{F}_T$ may be identified with the comma category $T \downarrow Id_{\mathcal{D}}$. Therefore there are 3 functors:

$$p_1 : \mathcal{D} \int \mathbf{F}_T \rightarrow \mathcal{C}, p_2 : \mathcal{D} \int \mathbf{F}_T \rightarrow \mathcal{D}, s : \mathcal{C} \rightarrow \mathcal{D} \int \mathbf{F}_T$$

with

$$p_1([T(c) \xrightarrow{\varphi} d]) = c, p_2([T(c) \xrightarrow{\varphi} d]) = d, s(c) = [T(c) = T(c)].$$

We have:

1.4.5. **Lemma.** *s is left adjoint to p_1 . Hence s and p_1 induce quasi-inverse homotopy equivalences $N(\mathcal{C}) \simeq N(\mathcal{D} \int \mathbf{F}_T)$.* \square

From Theorem 1.4.3 and Lemma 1.3.5, we deduce

1.4.6. **Corollary** (cf. [4, App. II]). *There is a spectral sequence*

$$E_{p,q}^2 = H_p(\mathcal{D}, H_q(\mathbf{F}_T, A)) \Rightarrow H_{p+q}(\mathcal{C}, A)$$

for any abelian group A . \square

2. A LONG HOMOLOGY EXACT SEQUENCE

2.1. Spectral sequences and exact couples.

2.1.1. Let \mathcal{T} be a triangulated category with countable direct sums, and let $C_0 \xrightarrow{i_1} \dots \xrightarrow{i_n} C_n \xrightarrow{i_{n+1}} \dots$ be a sequence of objects of \mathcal{T} . Let C be a homotopy colimit (mapping telescope) of the C_n [1]. Let $H : \mathcal{T} \rightarrow \mathcal{A}$ be a (co)homological functor to some abelian category \mathcal{A} : we assume that H commutes with countable direct sums. (Alternately, we could refuse infinite direct sums and assume that i_n is an isomorphism for n large enough.) To get an associated spectral sequence, the simplest is the technique of exact couples [7, pp. 152–153]: for each n , choose a cone $C_{n/n-1}$ of f_n , so that the exact triangles

$$C_{p-1} \xrightarrow{i_p} C_p \xrightarrow{j_p} C_{p/p-1} \xrightarrow{k_p} C_{p-1}[1]$$

yield long homology exact sequences

$$\dots H_n(C_{p-1}) \xrightarrow{i_{p,n}} H_n(C_p) \xrightarrow{j_{p,n}} H_n(C_{p/p-1}) \xrightarrow{k_{p,n}} H_{n-1}(C_{p-1}) \dots$$

where $H_n(X) := H(X[-n])$. The exact couple defined by

$$D_{p,q} = H_{p+q}(C_p), \quad E_{p,q} = H_{p+q}(C_{p/p-1})$$

and the relevant i, j, k define a spectral sequence abutting to $H_{p+q}(C)$. The E^1 -terms of this spectral sequence are simply $E_{p,q}^1 = E_{p,q}$, and $d_{p,q}^1 = j_{p-1,p+q-1}k_{p,p+q}$.

Here is a more concrete description of this differential:

2.1.2. Lemma. *Let $C_{p/p-2}$ be a cone for $i_p i_{p-1}$, so that we may obtain a commutative diagram*

$$\begin{array}{ccccccc}
 C_{p-2} & \xlongequal{\quad} & C_{p-2} & & & & \\
 i_{p-1} \downarrow & & i_p i_{p-1} \downarrow & & & & \\
 C_{p-1} & \xrightarrow{i_p} & C_p & \xrightarrow{j_p} & C_{p/p-1} & \xrightarrow{k_p} & C_{p-1}[1] \\
 j_{p-1} \downarrow & & \downarrow & & = \downarrow & & j_{p-1}[1] \downarrow \\
 C_{p-1/p-2} & \xrightarrow{\bar{i}_p} & C_{p/p-2} & \xrightarrow{\bar{j}_p} & C_{p/p-1} & \xrightarrow{\bar{k}_p} & C_{p-1/p-2}[1]
 \end{array}$$

of exact triangles (from the suitable axiom of triangulated categories). Then $d_{p,q}^1$ is the boundary map $\bar{k}_{p,n}$ in the long exact sequence

$$\begin{aligned}
 \dots H_{p+q}(C_{p-1/p-2}) & \xrightarrow{\bar{i}_{p,n}} H_{p+q}(C_{p/p-2}) \\
 & \xrightarrow{\bar{j}_{p,n}} H_{p+q}(C_{p/p-1}) \xrightarrow{\bar{k}_{p,n}} H_{p+q-1}(C_{p-1/p-2}) \dots
 \end{aligned}$$

Proof. The diagram shows that $\bar{k}_p = j_{p-1}[1] \circ k_p$. \square

2.1.3. In the usual case of a filtered complex $C_p = F_p C$, we may of course choose $C_{p/p-1} = F_p C / F_{p-1} C$ and $C_{p/p-2} = F_p C / F_{p-2} C$.

2.1.4. Let $\mathcal{Q}_0 \xrightarrow{T_1} \mathcal{Q}_1 \xrightarrow{T_2} \dots \rightarrow \mathcal{Q}_n \rightarrow \dots$ be a sequence of categories and functors. Let $\mathcal{Q} = \varinjlim \mathcal{Q}_n$. (Since there are no natural transformations involved this is a naïve colimit, defined objectwise and morphismwise.) Considering the corresponding sequence of chain complexes of nerves

$$C_*(\mathcal{Q}_0) \xrightarrow{i_1} C_*(\mathcal{Q}_1) \dots$$

we get from the yoga of 2.1.1 a spectral sequence abutting to $H_*(\mathcal{Q})$ (possibly with coefficients).

If the functors T_n are faithful and injective on objects, the maps i_n are injective and we are in the simpler situation of 2.1.3.

2.2. Unreduced and reduced homology.

2.2.1. Let (X, x) be a pointed simplicial set. The *reduced homology* of X with coefficients in an abelian group A is

$$\begin{aligned}
 \tilde{H}_i(X, A) &= H_i(X, x; A) := \text{Coker}(H_i(x, A) \rightarrow H_i(X, A)) \\
 &= H_i(X, A) \text{ if } i \neq 0.
 \end{aligned}$$

This definition apparently depends on the choice of x : if we don't want to choose a basepoint, we may alternately define

$$\tilde{H}_i(X, A) = \text{Ker}(H_i(X, A) \rightarrow H_i(*, A)).$$

Any choice of $x \in X_0$ will split the map $X \rightarrow *$, realising $\tilde{H}_i(X, A)$ as the above-described direct summand of $H_i(X, A)$.

A homotopy cofibre sequence $X \rightarrow Y \rightarrow Z$ is equivalent to a homotopy cocartesian square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & Z \end{array}$$

hence the corresponding long exact homology sequence may be written (via Mayer-Vietoris!) as

$$\dots H_i(X, A) \rightarrow H_i(Y, A) \rightarrow \tilde{H}_i(Z, A) \rightarrow H_{i-1}(X, A) \rightarrow \dots$$

2.2.2. Suppose $\mathbf{F} : \mathcal{D} \rightarrow \mathbf{sSet}$ is a functor. We then define

$$C_*(\mathcal{C}, \tilde{\mathbf{F}}; A) = \text{Ker}(C_*(\mathcal{C}, \mathbf{F}) \rightarrow C_*(\mathcal{C})) \otimes A$$

where the last map is induced by the natural transformation $\mathbf{F} \rightarrow *$, and

$$H_i(\mathcal{C}, \tilde{\mathbf{F}}; A) = H_i(C_*(\mathcal{C}, \tilde{\mathbf{F}}; A)).$$

(Here the map $\mathbf{F} \rightarrow *$ is not necessarily split, so we have to be more careful. We think of $\tilde{\mathbf{F}}$ as a desuspension of the homotopy cofibre of $\mathbf{F} \rightarrow *$.)

More generally, if $\mathbf{F} \rightarrow \mathbf{G}$ is a morphism of functors, we define

$$C_*(\mathcal{C}, \mathbf{F} \rightarrow \mathbf{G}; A) = \text{cone}(C_*(\mathcal{C}, \mathbf{F}) \rightarrow C_*(\mathcal{C}, \mathbf{G}))[-1].$$

2.2.3. Suppose $\mathbf{F} : \mathcal{D} \rightarrow \mathbf{Cat}$ is a functor. We have the projection $\mathbf{F} \rightarrow *$, where $*$ is the constant functor with values the category with 1 object and 1 morphism. As in 2.2.2 we define

$$\begin{aligned} C_*(\mathcal{C}, \tilde{\mathbf{F}}; A) &= \text{Ker}(C_*(\mathcal{C}, \mathbf{F}) \rightarrow C_*(\mathcal{C})) \otimes A \\ H_i(\mathcal{C}, \tilde{\mathbf{F}}; A) &= H_i(C_*(\mathcal{C}, \tilde{\mathbf{F}}; A)). \end{aligned}$$

2.3. Cellular functors.

2.3.1. **Lemma.** *Let \mathbf{F}_T be as in 1.4.4. Suppose T fully faithful. Then, for any $c \in \mathcal{C}$, $\mathbf{F}_T(T(c))$ has a final object.*

Proof. Such a final object is given by $[T(c) = T(c)]$. □

2.3.2. **Definition.** Let $T : \mathcal{C} \rightarrow \mathcal{D}$ be a functor as in 1.4.4. We say that T is *cellular* if

- T is fully faithful.
- For any $d \in \mathcal{D} - \mathcal{C}$ and any $c \in \mathcal{C}$, $\mathcal{D}(d, c) = \emptyset$.

2.3.3. Definition. A (naturally) commutative square of categories and functors is *homotopy cocartesian* if it is after applying the nerve functor.

2.3.4. Proposition. Let $\mathcal{D} - \mathcal{C}$ be the full subcategory of \mathcal{D} given by the objects not in \mathcal{C} . If T is cellular, the naturally commutative diagram of categories

$$\begin{array}{ccc} (\mathcal{D} - \mathcal{C}) \int \mathbf{F}_T & \xrightarrow{p} & \mathcal{C} \\ \varepsilon \downarrow & & T \downarrow \\ \mathcal{D} - \mathcal{C} & \xrightarrow{\iota} & \mathcal{D} \end{array}$$

is homotopy cocartesian, where ε is the augmentation (see §1.4.2), p is induced by the first projection p_1 of Lemma 1.4.5 and ι is the inclusion.

Proof. Note first the natural transformation

$$\begin{aligned} u : T \circ p &\Rightarrow \iota \circ \varepsilon \\ u_{[T(c) \xrightarrow{f} d]} &= f \end{aligned}$$

which explains “naturally commutative” (here in the weak sense). By Theorem 1.4.3 and Lemma 1.4.5, it suffices to prove that the (commutative) diagram of simplicial sets

$$\begin{array}{ccc} \delta N(\mathcal{D} - \mathcal{C}, \mathbf{F}_T) & \longrightarrow & \delta N(\mathcal{D}, \mathbf{F}_T) \\ \downarrow & & \downarrow \\ N(\mathcal{D} - \mathcal{C}) & \longrightarrow & N(\mathcal{D}) \end{array}$$

is homotopy cocartesian. The horizontal maps are cofibrations, hence it suffices to show that the map induced on the cofibres is a weak equivalence. We may consider the diagram as the diagonal of a diagram of bisimplicial sets:

$$\begin{array}{ccc} N(\mathcal{D} - \mathcal{C}, N(\mathbf{F}_T)) & \longrightarrow & N(\mathcal{D}, N(\mathbf{F}_T)) \\ \downarrow & & \downarrow \\ N(\mathcal{D} - \mathcal{C}, N(*)) & \longrightarrow & N(\mathcal{D}, N(*)). \end{array}$$

In the cofibres, the cells indexed by $d_0 \rightarrow \cdots \rightarrow d_n$, $d_i \in \mathcal{D} - \mathcal{C}$, are crushed down to a common point. By the assumption, $\mathcal{D}(\mathcal{D} - \mathcal{C}, \mathcal{C}) = \emptyset$, hence the remaining cells must all be indexed by chains of the form $c_0 \rightarrow \cdots$, with $c_0 \in \mathcal{C}$. But Lemma 2.3.1 then shows that the hypothesis of Lemma 1.3.3 b) is satisfied; hence the map on the cofibres becomes a weak equivalence after taking the diagonal, as desired. \square

2.3.5. Theorem. *Let $T : \mathcal{C} \rightarrow \mathcal{D}$ be a cellular functor. Then the mapping cone of*

$$C_*(T) : C_*(\mathcal{C}) \rightarrow C_*(\mathcal{D})$$

is quasi-isomorphic to $C_(\mathcal{D} - \mathcal{C}, \tilde{\mathbf{F}}_T)[1]$ (cf. 2.2.2). In particular, we have a long exact sequence*

$$\begin{aligned} \cdots \rightarrow H_i(\mathcal{D} - \mathcal{C}, \tilde{\mathbf{F}}_T; A) &\rightarrow H_i(\mathcal{C}, A) \\ &\rightarrow H_i(\mathcal{D}, A) \rightarrow H_{i-1}(\mathcal{D} - \mathcal{C}, \tilde{\mathbf{F}}_T; A) \rightarrow \cdots \end{aligned}$$

for any abelian group A .

Proof. This follows from Proposition 2.3.4 and Theorem 1.4.3. \square

2.4. Cellular filtrations.

2.4.1. Theorem. *Let $\mathcal{Q}_1 \rightarrow \mathcal{Q}_2 \rightarrow \cdots \rightarrow \mathcal{Q}_n \rightarrow \cdots \rightarrow \mathcal{Q}$ be a sequence of categories. We assume:*

- *The functors $T_n : \mathcal{Q}_{n-1} \rightarrow \mathcal{Q}_n$ are cellular (2.3.2).*
- *$\mathcal{Q} = \varinjlim \mathcal{Q}_n$.*

Write \mathbf{F}_n for \mathbf{F}_{T_n} . Then, for any abelian group A , there is a spectral sequence of homological type

$$E_{p,q}^1 = H_{p+q-1}(\mathcal{Q}_p - \mathcal{Q}_{p-1}, \tilde{\mathbf{F}}_p; A) \Rightarrow H_{p+q}(\mathcal{Q}, A).$$

Proof. This is the spectral sequence of 2.1.4, taking Theorem 2.3.5 into account. \square

3. THE RANK SPECTRAL SEQUENCE

3.1. K -theory of schemes. The first example of application of Theorem 2.4.1 is to Quillen's Q -construction $\mathcal{Q}(X)$ on the exact category of locally free sheaves of finite rank over a scheme X . Let $\mathcal{Q}_n = \mathcal{Q}_n(X)$ be the full subcategory of $\mathcal{Q}(X)$ consisting of locally free sheaves of rank $\leq n$. Then the assumptions of Theorem 2.4.1 are satisfied because, in $\mathcal{Q}(X)$, there are no morphisms from a locally free sheaf of rank n to a locally free sheaf of rank $< n$. The resulting spectral sequence may be called the *rank spectral sequence* (for the homology of $\mathcal{Q}(X)$).

Note that $\mathcal{Q}_n - \mathcal{Q}_{n-1}$ is a groupoid, hence we get

$$(3.1) \quad E_{p,q}^1 = \bigoplus_{E_\alpha} H_{p+q-1}(\text{Aut}(E_\alpha), \tilde{\mathbf{F}}_p)$$

where E_α runs through the set of isomorphism classes of locally free sheaves of rank p . We took coefficients \mathbf{Z} , for simplicity.

3.2. K' -theory of integral schemes. Let X be an integral scheme¹, with function field K . If $\eta = \operatorname{Spec} K$ is the generic point of X , we have the inclusion $j = \eta \rightarrow X$. If E is a sheaf of \mathcal{O}_X -modules, we write E_K for j^*E .

3.2.1. Definition. a) A coherent sheaf E on X is *torsion-free* if the map $E \rightarrow j_*E_K$ is a monomorphism.

b) A subsheaf E' of a coherent sheaf E is *pure* if E/E' is torsion-free.

3.2.2. Inside the exact category of coherent sheaves, the full subcategory of torsion-free sheaves is closed under extensions and subobjects. A monomorphism $E' \rightarrow E$ between torsion-free sheaves is admissible (within the exact category of torsion-free sheaves) if and only if E' is pure in E .

3.2.3. Lemma. Let $\mathcal{Q}^{\operatorname{coh}}(X)$ be Quillen's Q -construction on the category of coherent sheaves of \mathcal{O}_X -Modules, and let $\mathcal{Q}^{\operatorname{tf}}(X)$ be the full subcategory of torsion-free sheaves. Then the inclusion $\mathcal{Q}^{\operatorname{tf}}(X) \rightarrow \mathcal{Q}^{\operatorname{coh}}(X)$ is a weak equivalence.

Proof. The conditions of the resolution theorem [9, Th. 3] are verified since any locally free sheaf is torsion-free. \square

3.2.4. Proposition. Let E be a (coherent) torsion-free sheaf on X , with generic fibre E_K . Then the map

$$F \mapsto F_K$$

defines a bijection from the set $\operatorname{Gr}(E)$ of pure subsheaves of E to the set $\operatorname{Gr}(E_K)$ of subvector spaces of E_K .

Proof. Let V be a sub-vector space of $j^*E = E_K$. Define

$$E \cap V = E \times_{j_*j^*E} j_*V.$$

Then $E \cap V \in \operatorname{Gr}(E)$, because the map $E/(E \cap V) \rightarrow j_*j^*(E/(E \cap V))$ is a monomorphism (by definition of $E \cap V$). So we have two maps:

$$j^* : \operatorname{Gr}(E) \rightarrow \operatorname{Gr}(E_K); \quad E \cap - : \operatorname{Gr}(E_K) \rightarrow \operatorname{Gr}(E).$$

We have

$$(E \cap V)_K = j^*(E \times_{j_*j^*E} j_*V) = j^*E \times_{j_*j^*E} j^*j_*V = j^*E \times_{j^*E} V = V$$

so that $j^* \circ (E \cap -) = \operatorname{Id}$. On the other hand, if F is a pure subsheaf of E , then $F \subseteq E \cap j^*F$, and $j^*E = j^*(E \cap j^*F)$, hence (by exactness of j^*), $j^*(E \cap j^*F/F) = 0$. Thus $(E \cap j^*F)/F$ is a torsion subsheaf of

¹At least if X is Noetherian this restriction is not essential since $K'_*(X_{\operatorname{red}}) \xrightarrow{\sim} K'_*(X)$ and X_{red} is the disjoint union of its irreducible components.

the torsion-free sheaf E/F , hence is 0, and our two maps are inverse to each other. \square

3.2.5. We write $\mathcal{Q}_n^{\text{tf}}(X)$ for the full subcategory of $\mathcal{Q}^{\text{tf}}(X)$ of torsion-free sheaves E such that $\dim_K E_K \leq n$. We get another rank spectral sequence

$$(3.2) \quad E_{p,q}^1 = \bigoplus_{E_\alpha} H_{p+q-1}(\text{Aut}(E_\alpha), \tilde{\mathbf{F}}_p) \Rightarrow H_{p+q}(\mathcal{Q}^{\text{coh}}(X))$$

cf. Lemma 3.2.3.

3.2.6. **Corollary.** *Let $E \in \mathcal{Q}_n^{\text{tf}}(X)$, with generic fibre $E_K \in \mathcal{Q}_n(K)$. Then the functor*

$$j^* : \mathcal{Q}_{n-1}^{\text{tf}}(X) \downarrow E \rightarrow \mathcal{Q}_{n-1}(K) \downarrow E_K$$

is an isomorphism of categories.

Proof. These categories are those of proper admissible subsheaves of E and j^*E , with morphisms the inclusions (thanks to the definition of morphisms in $\mathcal{Q}^{\text{tf}}(X)$ and $\mathcal{Q}(K)$). Thus they are ordered sets, and the result directly follows from Proposition 3.2.4. \square

3.2.7. *Example.* Let X be an integral Dedekind scheme (= noetherian, regular of Krull dimension ≤ 1), with function field K . As is well-known, a coherent sheaf F over a Dedekind scheme is torsion-free if and only if it is locally free. (Being locally free is checked locally; a module of finite type over a discrete valuation ring is free if and only if it is torsion-free.) Thus the above generalises the remark of [10, pp. 191–192].

3.3. The Tits building.

3.3.1. In Corollary 3.2.6, suppose $n \geq 2$. By [10, Prop. p. 188], the classifying space of the ordered set $\mathcal{Q}_{n-1}(K) \downarrow E_K = J(E_K)$ is $GL(E_K)$ -weakly equivalent to the suspension of nerve of the Tits building of E_K , which in turn is weakly equivalent to a wedge of $(n-2)$ -spheres by the Solomon-Tits theorem [10, Th. 2 p. 180]. Hence $(N_n)_{|E}$ is $\text{Aut}(E)$ -weakly equivalent to a wedge of $(n-1)$ -spheres.

3.3.2. In Corollary 3.2.6, suppose $n = 1$. Then $\mathcal{Q}_{n-1}(K) \downarrow E_K$ has one element: $\{0\}$. Hence the conclusion of 3.3.1 is still true.

3.3.3. **Theorem.** *If X is an integral scheme, then the E^1 -terms of the rank spectral sequence (3.2) (with \mathbf{Z} -coefficients) are*

$$E_{p,q}^1 = \bigoplus_{E_\alpha} H_q(\text{Aut}(E_\alpha), st(E_\alpha))$$

where E_α runs through the isomorphism classes of locally free sheaves of rank p , and $st(E_\alpha) = \pi_{p-1}((\mathbf{F}_p)_{|E_\alpha})$ is the Steinberg module of j^*E_α .

Proof. This follows from Corollary 3.2.6, 3.3.1, 3.3.2 and Lemma 1.3.5. \square

3.3.4. *Remark.* By an argument of Vogel, the exact sequences from Theorem 2.3.5 then coincide with those of Quillen in [10, Th. 3 p. 181]. In general, consider a map $f : E \rightarrow B$ whose homotopy fibre F has the homotopy type of a bouquet of n -spheres, with $n > 0$. So the Leray-Serre spectral sequence yields a long exact sequence

$$(3.3) \quad \cdots \rightarrow H_p(E) \rightarrow H_p(B) \rightarrow H_{p-n-1}(B, H_n(F)) \rightarrow H_{p-1}(E) \rightarrow \cdots$$

If C is the homotopy cofibre of f , we have another long exact sequence

$$(3.4) \quad \cdots \rightarrow H_p(E) \rightarrow H_p(B) \rightarrow \tilde{H}_p(C) \rightarrow H_{p-1}(E) \rightarrow \cdots$$

and we want to know that the two sequences coincide.

Here is Vogel's argument. Let E' be the mapping cone of f and CF the cone over F , so that we have a fibration of pairs

$$\begin{array}{ccc} (CF, F) & \longrightarrow & (E', E) \\ & & \downarrow \\ & & (B, B). \end{array}$$

Since $H_q(CF, F) = \begin{cases} H_n(F) & \text{for } q = n+1 \\ 0 & \text{else} \end{cases}$, the Leray-Serre spectral sequence for the pair

$$H_p(B, H_q(CF, F)) \Rightarrow H_{p+q}(E', E)$$

yields isomorphisms

$$\tilde{H}_p(C) \simeq H_p(E', E) \simeq H_{p-n-1}(B, H_{n+1}(CF, F)) \simeq H_{p-n-1}(B, H_n(F))$$

which commute with the differentials of (3.3) and (3.4) by functoriality.

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